

Consistency Decision

Michael Pfender*

April 2014

last revised May 16, 2014

Abstract

The consistency formula for set theory can be stated in terms of the free-variables theory of primitive recursive maps. Free-variable p.r. predicates are decidable by set theory, main result here, built on recursive evaluation of p.r. map codes and soundness of that evaluation in set theoretical frame: internal p.r. map code equality is evaluated into set theoretical equality. So the free-variable consistency predicate of set theory is decided by set theory, ω -consistency assumed. By Gödel's second incompleteness theorem on undecidability of set theory's consistency formula by set theory under assumption of this ω -consistency, classical set theory turns out to be ω -inconsistent.

*michael.pfender@alumni.tu-berlin.de

Contents

1	Primitive recursive maps	2
2	PR code sets and evaluation	4
3	PR soundness within set theory	6
4	PR-predicate decision	11

1 Primitive recursive maps

Define the theory **PR** of objects and p.r. maps as follows recursively as a subsystem of **set** theory **T** :

- the objects
 $\mathbb{1} = \{0\}, \mathbb{N}, \mathbb{N} \times \mathbb{N}, \dots, A, \dots, B, A \times B$ etc.
- the map constants
 $0 : \mathbb{1} \rightarrow \mathbb{N}$ (*zero*), $s = s(n) = n + 1$ (*successor*), $\text{id}_A : A \rightarrow A$ (*identities*), $\Pi : A \rightarrow \mathbb{1}$ (*terminal* maps), $l = l(a, b) = a : A \times B \rightarrow A$, $r = r(a, b) = b : A \times B \rightarrow B$ (*left and right projections*);
- closure against (associative) map *composition*,
 $g \circ f = (g \circ f)(a) = g(f(a)) : A \rightarrow B \rightarrow C$;
- closure against forming the *induced* map $(f, g) = (f, g)(c) = (f(c), g(c)) : C \rightarrow A \times B$ into a product, for given components $f : C \rightarrow A$, $g : C \rightarrow B$,
 $l \circ (f, g) = f$, $r \circ (f, g) = g$;

- closure against forming the *iterated* map

$$\begin{aligned}
f^{\S} &= f^{\S}(a, n) = f^n(a) : A \times \mathbb{N} \rightarrow A, \\
f^0(a) &= \text{id}_A(a) = a, \\
f^{sn}(a) &= f^{\S}(a, sn) = (f \circ f^{\S})(a, n) = f(f^n(a, n)).
\end{aligned}$$

Furthermore **PR** is to inherit from **T** uniqueness of the *initialised iterated*, in order to inherit uniqueness in the following *full schema of primitive recursion*:

$$\begin{array}{l}
g = g(a) : A \rightarrow B \text{ (initialisation),} \\
h = h((a, n), b) : (A \times \mathbb{N}) \times B \rightarrow B \text{ (step)} \\
\text{(pr)} \quad \frac{}{f = f(a, n) : A \times \mathbb{N} \rightarrow B,} \\
f(a, 0) = g(a) \\
f(a, sn) = h((a, n), f(a)) \\
+ \text{uniqueness of such p. r. defined map } f.
\end{array}$$

This schema allows in particular construction of for loops,

for $i := 1$ to n do . . od

as for verification if a given text (code) is an (arithmetised) *proof* of a given coded assertion, Gödel's p. r. formula 45. xBy , x ist *Beweis* von y .

(Formel 46. *Bew* $y = \exists xBy$, x is *provable*, is not p. r.)

2 PR code sets and evaluation

The *map code set*—set of gödel numbers—we want to *evaluate* is $\mathbf{PR} = \bigcup_{A,B} [A, B] \subset \mathbb{N}$ in \mathbf{T} , $[A, B] = [A, B]_{\mathbf{PR}}$ the set of p. r. map codes from A to B .

Together with evaluation on suitable arguments it is recursively defined as follows:

- Basic map constants ba in \mathbf{PR} :

$$- \ulcorner 0 \urcorner \in [\mathbb{1}, \mathbb{N}] \subset \mathbf{PR} \text{ (zero),}$$

$$ev(\ulcorner 0 \urcorner, 0) = 0,$$

$$\ulcorner s \urcorner \in [\mathbb{N}, \mathbb{N}] \text{ (successor),}$$

$$ev(\ulcorner s \urcorner, n) = s(n) = n + 1,$$

$$- \text{ For an object } A \ulcorner \text{id}_A \urcorner \in [A, A] \text{ (identity),}$$

$$ev(\ulcorner \text{id}_A \urcorner, a) = \text{id}_A(a) = a,$$

$$\ulcorner \Pi_A \urcorner \in [A, \mathbb{1}] \text{ (terminal map),}$$

$$ev(\ulcorner \Pi_A \urcorner, a) = \Pi_A(a) = 0.$$

$$- \text{ for objects } A, B \ulcorner l_{A,B} \urcorner \in [A \times B, A] \text{ (left projection),}$$

$$ev(\ulcorner l_{A,B} \urcorner, (a, b)) = l_{A,B}(a, b) = a,$$

$$\ulcorner r_{A,B} \urcorner \in [A \times B, B] \text{ (right projection),}$$

$$ev(\ulcorner r_{A,B} \urcorner, (a, b)) = r_{A,B}(a, b) = b.$$

- For $u \in [A, B], v \in [B, C] : v \odot u \in [A, C]$
(internal composition),

$$ev(v \odot u, a) = ev(v, ev(u, a)).$$

- For $u \in [C, A], v \in [C, B] : \langle u; v \rangle \in [C, A \times B]$
(induced map code into a product),

$$ev(\langle u; v \rangle, c) = (ev(u, c), ev(v, c)).$$

- For $u \in [A, A] : u^\$ \in [A \times \mathbb{N}, A]$ (iterated map code),

$$ev(u^\$, 0) = \text{id}_A(a) = a,$$

$$ev(u^\$, sn) = ev(u, ev(u^\$, n)) \text{ (double recursion)}$$

This recursion *terminates* in set theory **T**, with correct results:

Objectivity Theorem: Evaluation ev is *objective*, i. e. for $f : A \rightarrow B$ in **PR** we have

$$ev(\ulcorner f \urcorner, a) = f(a).$$

Proof by substitution of codes of **PR** maps into code variables $u, v \in \text{PR} \subset \mathbb{N}$ in the above double recursive definition of evaluation, in particular:

- composition

$$\begin{aligned} ev(\ulcorner g \urcorner \odot \ulcorner f \urcorner, a) &= ev(\ulcorner g \urcorner, ev(\ulcorner f \urcorner, a)), \\ &= g(f(a)) = (g \circ f)(a) \end{aligned}$$

recursively, and

- iteration

$$\begin{aligned} ev(\ulcorner f^{\ulcorner \S \urcorner}, \langle a; sn \rangle) &= ev(\ulcorner f^{\ulcorner \neg \urcorner}, ev(\ulcorner f^{\ulcorner \S \urcorner}, \langle a; n \rangle)) \\ &= f(f^{\ulcorner \S \urcorner}(a, n)) = f(f^n(a)) = f^{sn}(a) \end{aligned}$$

recursively.

3 PR soundness within set theory

Notion $f =^{\mathbf{PR}} g$ of p.r. maps is externally p.r. enumerated, by complexity of (binary) deduction trees.

Internalising—*formalising*—gives an internal notion of PR equality,

$$u \dot{=}^k v \in \mathbf{PR} \times \mathbf{PR}$$

coming by k th internal equation *proved* by k th internal *deduction tree* dtree_k .

PR evaluation *soundness* theorem framed by set theory **T** : For p.r. theory **PR** with its internal notion of equality ‘ $\dot{=}$ ’ we have:

- (i) PR to **T** evaluation soundness:

$$\mathbf{T} \vdash u \dot{=} v \implies ev(u, x) = ev(v, x) \quad (\bullet)$$

Substituting in the above “concrete” **PR** codes into u resp. v , we get, by *objectivity* of evaluation ev :

- (ii) **T**-framed objective soundness of **PR** :

For p.r.maps $f, g : A \rightarrow B$

$$\mathbf{T} \vdash \ulcorner f^{\ulcorner \neg \urcorner} \dot{=} \ulcorner g^{\ulcorner \neg \urcorner} \implies f(a) = g(a).$$

- (iii) Specialising to case $f := \chi : A \rightarrow 2 = \{0, 1\}$ a p.r. predicate, and to $g := \text{true}$, we get
T-framed *logical soundness of PR* :

$$\mathbf{T} \vdash \exists k \text{Prov}_{\mathbf{PR}}(k, \ulcorner \chi \urcorner) \implies \forall x \chi(x) :$$

*If a p. r. predicate is—within **T**—**PR**-internally provable, then it holds in **T** for all of its arguments.*

Proof by primitive recursion on k , dtree_k the k th deduction tree of the theory, *proving* its root equation $u \dot{=}^k v$. These (argument-free) deduction trees are counted in lexicographical order.

Super Case of *equational* internal axioms, in particular

- associativity of (internal) composition:

$$\langle \langle w \odot v \rangle \odot u \rangle \dot{=} \langle w \odot \langle v \odot u \rangle \rangle \implies$$

$$\begin{aligned} \text{ev}(\langle \langle w \odot v \rangle \odot u, a \rangle) &= \text{ev}(\langle \langle w \odot v \rangle, \text{ev}(u, a) \rangle) \\ &= \text{ev}(w, \text{ev}(v, \text{ev}(u, a))) \\ &= \text{ev}(w, \text{ev}(\langle v \odot u \rangle, a)) = \text{ev}(w \odot \langle v \odot u \rangle, a). \end{aligned}$$

This **proves** assertion (•) in present *associativity-of-composition* case.

- Analogous **proof** for the other flat, equational cases, namely *reflexivity of equality, left and right neutrality* of identities, all substitution equations for the map constants, Godement's equations for the induced map:

$$l \odot \langle u; v \rangle \dot{=} u, \quad r \odot \langle u; v \rangle \dot{=} v,$$

as well as *surjective pairing*

$$\langle l \odot w; r \odot w \rangle \doteq w$$

and distributivity equation

$$\langle u; v \rangle \odot w \doteq \langle u \odot w; v \odot w \rangle$$

for composition with an induced.

- **proof** of (\bullet) for the last equational **case**, the

Iteration step, case of genuine iteration equation

$u^\$ \odot \langle \ulcorner \text{id} \urcorner \# \ulcorner s \urcorner \rangle \doteq u \odot u^\$$, $\#$ the internal cartesian product of map codes:

$$\mathbf{T} \vdash \text{ev} (u^\$ \odot \langle \ulcorner \text{id} \urcorner \# \ulcorner s \urcorner \rangle, \langle a; n \rangle) \quad (1)$$

$$= \text{ev} (u^\$, \text{ev} (\langle \ulcorner \text{id} \urcorner \# \ulcorner s \urcorner \rangle, \langle a; n \rangle))$$

$$= \text{ev} (u^\$, \langle a; sn \rangle)$$

$$= \text{ev} (u, \text{ev} (u^\$, \langle a; n \rangle))$$

$$= \text{ev} (u \odot u^\$, \langle a; n \rangle). \quad (2)$$

Proof of termination-conditioned inner soundness for the remaining genuine HORN case axioms, of form

$$u \doteq_i v \wedge u' \doteq_j v' \implies w \doteq_k w', \ i, j < k :$$

Transitivity-of-equality case

$$u \doteq_i v \wedge v \doteq_j w \implies u \doteq_k w :$$

Evaluate at argument $a \in A$ and get in fact

$$\begin{aligned}
\mathbf{T} &\vdash u \dot{=}_k w \\
&\implies ev(u, a) = ev(v, a) \wedge ev(v, a) = ev(w, a) \\
&\text{(by hypothesis on } u, v) \\
&\implies ev(u, a) = ev(w, a) : \\
&\text{transitivity export q.e.d. in this case.}
\end{aligned}$$

Compatibility case of composition with equality,

$$\begin{aligned}
u \dot{=} u' &\implies \langle v \odot u \rangle \dot{=} \langle v \odot u' \rangle : \\
ev(v \odot u, a) &= ev(v, ev(u, a)) = ev(v, ev(u', a)) \\
&= ev(v \odot u', a),
\end{aligned}$$

by hypothesis on $u \dot{=} u'$ and by Leibniz' substitutivity in \mathbf{T} , q.e.d. in this first compatibility case.

Case of composition with equality in second composition factor,

$$\begin{aligned}
v \dot{=} v' &\implies \langle v \odot u \rangle \dot{=} \langle v' \odot u \rangle : \\
ev(\langle v \odot u \rangle, x) &= ev(v, ev(u, x)) = ev(v', ev(u, x)) \quad (*) \\
&= ev(\langle v' \odot u \rangle, x).
\end{aligned}$$

(*) holds by $v \dot{=} v'$, induction hypothesis on v, v' , and Leibniz' substitutivity: same argument put into equal maps.

This proves soundness assertion (\bullet) in this 2nd compatibility case.

(Redundant) Case of **compatibility** of forming the induced map, with equality, is analogous to compatibilities above,

even easier, since the two map codes concerned are independent from each other what concerns their domains.

(Final) Case of Freyd’s (internal) **uniqueness** of the *initialised iterated*, is **case**

$$\begin{aligned} & \langle w \odot \langle \ulcorner \text{id} \urcorner; \ulcorner 0 \urcorner \odot \ulcorner \Pi \urcorner \rangle \dot{=}_i u \rangle \\ & \wedge \langle w \odot \langle \ulcorner \text{id} \urcorner \# \ulcorner s \urcorner \rangle \dot{=}_j \langle v \odot w \rangle \rangle \\ & \implies w \dot{=}_k v^{\$} \odot \langle u \# \ulcorner \text{id} \urcorner \rangle \end{aligned}$$

Comment: w is here an internal *comparison candidate* fullfilling the same internal p.r. equations as the *initialised iterated* $\langle v^{\$} \odot \langle u \# \ulcorner \text{id} \urcorner \rangle \rangle$. It should be – **is:** *soundness* – evaluated equal to the latter, on $A \times \mathbb{N}$.

Soundness **assertion** (\bullet) for the present Freyd’s *uniqueness case* recurs on $\dot{=}_i, \dot{=}_j$ turned into predicative equations ‘=’, these being already deduced, by hypothesis on $i, j < k$. Further ingredients are transitivity of ‘=’ and established properties of evaluation *ev*.

So here is the remaining – inductive – **proof**, prepared by

$$\mathbf{T} \vdash \text{ev}(w, \langle a; 0 \rangle) = \text{ev}(u; a) \quad (\bar{0})$$

as well as

$$\begin{aligned} \text{ev}(w, \langle a; sn \rangle) &= \text{ev}(w \odot \langle \ulcorner \text{id} \urcorner \# \ulcorner s \urcorner \rangle, \langle a; n \rangle) \\ &= \text{ev}(v \odot w, \langle a; n \rangle), \end{aligned} \quad (\bar{s})$$

the same being true for $w' := v^{\$} \odot \langle u \# \ulcorner \text{id} \urcorner \rangle$ in place of w , once more by (characteristic) double recursive equations for *ev*, this time with respect to the *initialised internal iterated* itself.

$(\bar{0})$ and (\bar{s}) put together for both then show, by induction on *iteration count* $n \in \mathbb{N}$ —all other free variables u, v, w, a together form the *passive parameter* for this induction—*soundness* assertion (\bullet) for this *Freyd's uniqueness* case, namely

$$\mathbf{T} \vdash \text{ev}(w, \langle a; n \rangle) = \text{ev}(v^{\$} \odot \langle u \#^{\ulcorner \text{id} \urcorner}, \langle a; n \rangle).$$

Induction runs as follows:

Anchor $n = 0$:

$$\text{ev}(w, \langle a; 0 \rangle) = \text{ev}(u, a) = \text{ev}(w', \langle a; 0 \rangle),$$

step:

$$\begin{aligned} \text{ev}(w, \langle a; n \rangle) &= \text{ev}(w', \langle a; n \rangle) \implies \\ \text{ev}(w, \langle a; sn \rangle) &= \text{ev}(v, \text{ev}(w, \langle a; n \rangle)) \\ &= \text{ev}(v, \text{ev}(w', \langle a; n \rangle)) = \text{ev}(w', \langle a; sn \rangle), \end{aligned}$$

q. e. d.

4 PR-predicate decision

We consider here **PR** predicates for decidability by **set** theorie(s) **T**. Basic tool is **T**-framed soundness of **PR** just above, namely

$$\chi = \chi(a) : A \rightarrow 2 \text{ PR predicate}$$

$$\mathbf{T} \vdash \exists k \text{Prov}_{\mathbf{PR}}(k, \ulcorner \chi \urcorner) \implies \forall a \chi(a).$$

Within \mathbf{T} define for $\chi : A \rightarrow \mathbb{2}$ out of \mathbf{PR} a partially defined (alleged, individual) μ -recursive *decision* $\nabla\chi : \mathbb{1} \rightarrow \mathbb{2}$ by first fixing *decision domain*

$$D = D_\chi := \{k \in \mathbb{N} : \neg\chi(\text{ct}_A(k)) \vee \text{Prov}_{\mathbf{PR}}(k, \ulcorner\chi\urcorner)\},$$

$\text{ct}_A : \mathbb{N} \rightarrow A$ (retractive) Cantor count of A ; and then, with (partial) recursive $\mu D : \mathbb{1} \rightarrow \mathbb{N}$ within \mathbf{T} :

$$\nabla\chi \stackrel{\text{def}}{=} \left\{ \begin{array}{l} \text{false if } \neg\chi(\text{ct}_A(\mu D)) \\ \quad (\text{counterexample}), \\ \text{true if } \text{Prov}_{\mathbf{PR}}(\mu D, \ulcorner\chi\urcorner) \\ \quad (\text{internal proof}), \\ \perp \text{ (undefined) otherwise, i. e.} \\ \quad \text{if } \forall a \chi(a) \wedge \forall k \neg \text{Prov}_{\mathbf{PR}}(k, \ulcorner\chi\urcorner). \end{array} \right.$$

[This (alleged) decision is apparently μ -recursive within \mathbf{T} , even if apriori only partially defined.]

There is a first *consistency* problem with this **definition**: are the *defined* cases *disjoint*?

Yes, within frame theory \mathbf{T} which *soundly frames* theory \mathbf{PR} :

$$\mathbf{T} \vdash (\exists k \in \mathbb{N}) \text{Prov}_{\mathbf{PR}}(k, \ulcorner\chi\urcorner) \implies \forall a \chi(a).$$

We show now, that decision $\nabla\chi$ is *totally defined*, the undefined case does not arise, this for \mathbf{T} ω -consistent in Gödel's sense.

We have the following complete – metamathematical – case distinction on $D = D_\chi \subseteq \mathbb{N}$:

- **1st case**, termination: D has at least one (“total”) PR point $\mathbb{1} \rightarrow D \subseteq \mathbb{N}$, and hence

$$t = t_{\nabla\chi} =_{\text{by def}} \mu D = \min D : \mathbb{1} \rightarrow D$$

is a (total) p.r. point.

Subcases:

- **1.1**, negative (total) **subcase:**

$$\neg \chi \text{ct}_A(t) = \text{true}.$$

$$[\text{Then } \mathbf{T} \vdash \nabla\chi = \text{false}.]$$

- **1.2**, positive (total) **subcase:**

$$\text{Prov}_{\mathbf{PR}}(t, \ulcorner \chi \urcorner) = \text{true}.$$

$$[\text{Then } \mathbf{T} \vdash \nabla\chi = \text{true},$$

by **T**-framed objective soundness of **PR**.]

These two subcases are *disjoint*, disjoint here by **T**-framed soundness of theory **PR** which reads

$$\mathbf{T} \vdash \text{Prov}_{\mathbf{PR}}(k, \ulcorner \chi \urcorner) \implies \forall a \chi(a), \text{ } k \text{ free},$$

here in particular – substitute $t : \mathbb{1} \rightarrow \mathbb{N}$ into k free:

$$\pi \mathbf{R} \vdash \text{Prov}_{\mathbf{PR}}(t, \ulcorner \chi \urcorner) \implies \forall a \chi(a).$$

So furthermore, by this framed soundness, in present **subcase:**

$$\mathbf{T} \vdash \forall a \chi(a) \wedge \text{Prov}_{\mathbf{PR}}(t, \ulcorner \chi \urcorner). \quad (\bullet)$$

- **2nd case**, derived non-termination:

$$T \vdash D = \emptyset \equiv \{\mathbb{N} : \text{false}_{\mathbb{N}}\} \subset \mathbb{N}$$

$$[\text{then in particular } \mathbf{T} \vdash \forall a \neg \chi(a) = \text{false},$$

so $\mathbf{T} \vdash \forall a \chi(a)$ in this case],
and furthermore

$$\begin{aligned} \mathbf{T} &\vdash \forall k \neg \text{Prov}_{\mathbf{PR}}(k, \ulcorner \chi \urcorner), \text{ so} \\ \mathbf{T} &\vdash \forall a \chi(a) \wedge \forall k \neg \text{Prov}_{\mathbf{PR}}(k, \ulcorner \chi \urcorner) \quad (*) \end{aligned}$$

in this case.

- **3rd**, remaining, *ill case* is:

D (metamathematically) *has no (total) points* $\mathbb{1} \rightarrow D$,
but is nevertheless not empty.

Take in the above the (disjoint) union of **2nd subcase** of **1st case**, (\bullet) , and of **2nd case**, $(*)$, as new case. And formalise last, remaining case. Arrive at the following

Quasi-Decidability Theorem: each p.r. predicate $\chi : A \rightarrow \mathbb{2}$ gives rise within **set** theory \mathbf{T} to the following complete (metamathematical) case distinction:

- (a) $\mathbf{T} \vdash \forall a \chi(a)$ or else
- (b) $\mathbf{T} \vdash \neg \chi_{\text{ct}_A t} : \mathbb{1} \rightarrow D_\chi \rightarrow \mathbb{2}$
(*defined counterexample*), or else
- (c) $D = D_\chi$ *non-empty, pointless*, formally: in this **case** we would have within \mathbf{T} :

$$\begin{aligned} \mathbf{T} &\vdash \exists \hat{a} \in D, \\ &\text{and “nevertheless” for each p.r. point } p : \mathbb{1} \rightarrow \mathbb{N} \\ \mathbf{T} &\vdash p \notin D. \end{aligned}$$

We **rule out** the latter – general – possibility of a *non-empty* predicate *without p.r. points*, for frame theory \mathbf{T} by

gödelian **assumption** of ω -consistency. In fact it rules out above instance of ω -inconsistency: all numerals $0, 1, 2, \dots$ are p.r. points. Hence it rules out – in *quasi-decidability* above – possibility (c) for decision domain $D = D_\chi \subseteq \mathbb{N}$ of decision operator ∇ for predicate $\chi : A \rightarrow 2$, and we get

Decidability theorem: Each free-variable p.r. predicate $\chi : A \rightarrow 2$ gives rise to the following **complete case distinction** by **set** theory **T** :

Under **assumption** of ω -consistency for **T** :

- $\mathbf{T} \vdash \forall a \chi(a)$ (*theorem*) or
- $\mathbf{T} \vdash (\exists a \in A) \neg \chi(a)$. (*counterexample*)

Now take here for predicate χ , **T**'s own free-variable p.r. consistency formula

$$\text{Con}_{\mathbf{T}} = \neg \text{Prov}_{\mathbf{T}}(k, \ulcorner \text{false} \urcorner) : \mathbb{N} \rightarrow 2,$$

and get, under **assumption** of ω -consistency for **T**, a **consistency decision** $\nabla_{\text{Con}_{\mathbf{T}}}$ for **T** by **T**.

This contradiction to (the postcedent of) Gödel's 2nd Incompleteness theorem shows that the **assumption** of ω -consistency for **set** theories **T** must fail:

Set theories **T** are ω -inconsistent.

This concerns all classical **set** theories as in particular **PM**, **ZF**, and **NGB**. The reason is ubiquity of formal quantification within these (arithmetical) theories.

Problem: Does it concern Peano Arithmetic either?

References

- [1] S. EILENBERG, C. C. ELGOT 1970: *Recursiveness*. Academic Press.
- [2] P. J. FREYD 1972: Aspects of Topoi. *Bull. Australian Math. Soc.* **7**, 1-76.
- [3] K. GÖDEL 1931: Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I. *Monatsh. der Mathematik und Physik* **38**, 173-198.
- [4] R. L. GOODSTEIN 1971: *Development of Mathematical Logic*, ch. 7: Free-Variable Arithmetics. Logos Press.
- [5] A. JOYAL 1973: Arithmetical Universes. Talk at Oberwolfach.
- [6] J. LAMBEK, P. J. SCOTT 1986: *Introduction to higher order categorical logic*. Cambridge University Press.
- [7] M. PFENDER 2008b: RCF 2: Evaluation and Consistency. arXiv:0809.3881v2 [math.CT]. Has a **gap**.
- [8] M. PFENDER 2013: *Arithmetical Foundations*, β version, arXiv 2013.
- [9] M. PFENDER 2014: *Arithmetical Foundations*, γ version, preprint no.??-2014 Mathematik TU Berlin.
- [10] M. PFENDER, M. KRÖPLIN, D. PAPE 1994: Primitive Recursion, Equality, and a Universal Set. *Math. Struct. in Comp. Sc.* **4**, 295-313.
- [11] L. ROMÀN 1989: Cartesian categories with natural numbers object. *J. Pure and Appl. Alg.* **58**, 267-278.

- [12] C. SMORYNSKI 1977: The Incompleteness Theorems. Part D.1 in BARWISE ed. 1977. *Handbook of Mathematical Logic*. North Holland.